Propagation of Errors

In the previous two classes, we have described the desired result and uncertainty of an experiment using the mean and standard deviation of a set of data. Now, it is necessary to see how these uncertainties affect calculations.

An example: Volume of a box

$$V_o = L_o * W_o * H_o$$

where L_o, W_o, and H_o are measured values.

If we know the uncertainty of the measurements,

$$\Delta L = L_o - L; \Delta W = W_o - W; \Delta H = H_o - H$$

we can estimate the uncertainty of V, ΔV.

ΔV is the uncertainty of each component times the contribution of each component.

$$\Delta V \approx \Delta L * (\partial V/\partial L)_{WoHo} + \Delta W * (\partial V/\partial W)_{LoHo} + \Delta H * (\partial V/\partial H)_{WoLo} \quad (1)$$

In this example, V = L * W * H. So, equation (1) gives

$$\Delta V/Vo \approx \Delta L/Lo + \Delta W/Wo + \Delta H/Ho$$

In most cases, the actual uncertainties are unknown. However, we do know the estimated error of the distribution of the data.

$$X = f(U, V, \ldots) \quad (2)$$

$$\overline{X} = f(\overline{U}, \overline{V}, \ldots) \quad (3)$$

$$X_i = f(U_i, V_i, \ldots) \quad (4)$$

$$\sigma_x^{\ 2} = \lim_{N\to\infty}\left[\frac{1}{N}\Sigma(X_i - \overline{X})^2\right] \quad (5)$$

As in equation (1), $(X_i - \overline{X})$ can be written as the sum of its components.

$$X_i - \overline{X} \approx (U_i - \overline{U})\left(\frac{\partial X}{\partial U}\right) + (V_i - \overline{V})\left(\frac{\partial X}{\partial V}\right) + \ldots \quad (6)$$

By combining equations (5) and (6):

$$\begin{aligned}\sigma_x{}^2 &\approx \lim_{N\to\infty}\left\{\frac{1}{N}\Sigma\left[\left(U_i-\overline{U}\right)\left(\frac{\partial X}{\partial U}\right)+\left(V_i-\overline{V}\right)\left(\frac{\partial X}{\partial V}\right)+\ldots\right]^2\right\} \\ &\approx \lim_{N\to\infty}\left\{\frac{1}{N}\Sigma\left[\left(U_i-\overline{U}\right)^2\left(\frac{\partial X}{\partial U}\right)^2+\left(V_i-\overline{V}\right)^2\left(\frac{\partial X}{\partial V}\right)^2+2\left(U_i-\overline{U}\right)\left(V_i-\overline{V}\right)\left(\frac{\partial X}{\partial U}\right)\left(\frac{\partial X}{\partial V}\right)\ldots\right]\right\}\end{aligned} \tag{7}$$

The first two terms can be expressed in terms of the ***variance*** of the components:

$$\sigma_U{}^2 = \lim_{N\to\infty}\left[\frac{1}{N}\Sigma(U_i-\overline{U})^2\right] \tag{8a}$$

$$\sigma_V{}^2 = \lim_{N\to\infty}\left[\frac{1}{N}\Sigma(V_i-\overline{V})^2\right] \tag{8b}$$

And in terms of the ***co-variance***:

$$\sigma_{UV}{}^2 \approx \lim_{N\to\infty}\left[\frac{1}{N}\Sigma\left(U_i-\overline{U}\right)\left(V_i-\overline{V}\right)\right] \tag{9}$$

So, equation (7) can be written as:

$$\sigma_X{}^2 \approx \sigma_U{}^2 * (\partial X/\partial U)^2 + \sigma_V{}^2 * (\partial X / \partial V)^2 + 2\sigma_{UV}{}^2 * (\partial X/\partial U)\ (\partial X / \partial V) + \ldots \tag{10}$$

This is the ***error propagation equation***.

When U and V are not related, the random fluctuations of the numbers will cancel themselves out and $\sigma_{UV}{}^2 \approx 0$.

Therefore:

$$\sigma_X{}^2 \approx \sigma_U{}^2 * (\partial X/\partial U)^2 + \sigma_V{}^2 * (\partial X / \partial V)^2 + \ldots \tag{11}$$

Equations (10) and (11) state the general case and can always be used to describe the uncertainty of a calculation. The following are more specific examples.

In the following examples, X is a function of U and V; a and b are constants.

Addition and Subtraction

$X = aU \pm bV$

$(\partial X/\partial U) = a$; $(\partial X/\partial V) = \pm\, b$

Substitute into Equation (10)
$\sigma_X^2 \approx \sigma_U^2 * a^2 + \sigma_V^2 * b^2 \pm 2\sigma_{UV}^2 * ab$ (12)

Multiplication and Division

$X = \pm aUV$
$(\partial X/\partial U) = \pm aV; \ (\partial X/\partial V) = \pm aU$

$\sigma_X^2 \approx \sigma_U^2 * a^2 V^2 + \sigma_V^2 * a^2 U^2 + 2\sigma_{UV}^2 * a^2 UV$

Simplify by dividing by $X^2 = a^2U^2V^2$.
$\sigma_X^2 / X^2 \approx \sigma_U^2 / U^2 + \sigma_V^2 / V^2 + 2\sigma_{UV}^2 / UV$ (13)

Similarly, if X is obtained through division,
$X = \pm aU/V$
$(\partial X/\partial U) = \pm a/V; \ (\partial X/\partial V) = \pm aU/V^2$

$\sigma_X^2 \approx \sigma_U^2 * a^2 / V^2 + \sigma_V^2 a^2 U^2 / V^4 - 2\sigma_{UV}^2 * a^2 U/V^3$

After simplifying, we have
$\sigma_X^2 / X^2 \approx \sigma_U^2 / U^2 + \sigma_V^2 / V^2 - 2\sigma_{UV}^2 / UV$ (14)

Exercises:

1. Find σ_X in terms of σ_U and σ_V. Assume that U and V are completely independent of each other (i.e. $\sigma_{UV}^2 = 0$).
   - (a) $X = ½ (U + V)$
   - (b) $X = ½ (U - V)$
   - (c) $X = 1/U^2$
   - (d) $X = UV^2$
   - (e) $X = U^2 + V^2$

2. Find σ_t for a substance whose activity is $10s^{-1}$, initial activity was $1000s^{-1}$, and natural life (τ) is 5 days. σ_A is $1s^{-1}$. Use the following equation for t: (Hint: consider initial activity and natural life to be constants in the following equation).
   $t = - \tau \ln (A/A_0)$

3. If a table is round and its diameter is determined to within 1% (relatively), how well is the area know (Hint: Find σ_A)? Would it be better to determine its radius to within 1%? (Hint: Use d = 1 m)